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RESEARCH ARTICLE

On the series expansion of the spatial SIS evolution operator

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For the spatial stochastic SIS model we consider the perturbative series expansion of the gap between the dominant and subdominant eigenvalues of the evolution operator. We compute explicitly the first terms of the series expansion of the gap.

Keywords: SIS model; contact process; evolution operator; series expansion; critical threshold

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1. Introduction

The characterization of the critical thresholds in epidemic models is probably the most important feature of the mathematical epidemiology research due to the drastic change of the disease spread on the critical threshold. The simple spatial stochastic SIS (Susceptible-Infected-Susceptible) epidemic model, also known as contact process, has a continuous phase transition from the absorbing state devoid of infected individuals to a nonequilibrium state of infectivity. The phase transition lines of the SIS model were recently characterized in pair approximation as particular case of the reinfection SIRI model [8, 10], and improves the rough qualitative behaviour in mean field approximation. In the phase transition the dominant eigenvalue of the evolution operator for the SIS model becomes degenerate, that occurs when the gap between the dominant and the subdominant eigenvalues vanishes. To study the gap value, series expansions in terms of the creation rate have been used [1, 2, 7]. This requires the formulation of the epidemic models in terms of creation and annihilation operators [3–5, 9, 11] starting from the master equation [6, 12]. The critical values follows from the series expansion, based on a perturbation ansatz and using a Padé analyses [1, 7].

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Here, we consider the spatial stochastic SIS epidemic model formulated via creation and annihilation operators. For the SIS model in one dimensional lattices, we deduce the perturbative series expansion of the gap between the dominant and subdominant eigenvalues of the evolution operator. The first terms of the series expansion of the gap are computed explicitly. Here, we do not assume the translation invariance of the lattice in contrast with the study made in [1].

2. The spatial stochastic SIS model

The stochastic SIS model is one of the best known epidemic models and one of the simplest. It describes the evolution of an infectious disease through a population of N individuals, which can be either infected or susceptible. This epidemic model is also known as the contact process because it describes an interacting-particle system on a regular lattice, where the particles are annihilated spontaneously and created catalytically. We consider that a particle is annihilated with rate α, usually $\alpha = 1$, and created with a rate β times the fraction of the nearest neighbours occupied sites. We use the state variables

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

to represent the site i whenever is empty or occupied or, in an epidemic context, when the individual i is susceptible or infected. The configuration of the lattice can be represented by

$$|\eta\rangle = |\eta_1\eta_2 \ldots \eta_N\rangle = |\eta_1\rangle \otimes |\eta_2\rangle \otimes \ \ldots \ \otimes |\eta_N\rangle \quad , \tag{2}$$

where $|\eta_i\rangle = |0\rangle$ or $|\eta_i\rangle = |1\rangle$ represents the state of each site i and N denotes the number of the sites.

2.1 The creation and annihilation operators

To describe the SIS epidemic model using the creation and annihilation operators, we define the following operators to act on each site of the lattice.

Definition 2.1. The creation operator, c_i^+, and the annihilation operator, c_i, are given by

$$c_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad c_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad . \tag{3}$$

It is obvious that these local operators applied to the state variables give

$$c_i^+ \, |0\rangle = |1\rangle \quad \text{and} \quad c_i \, |1\rangle = |0\rangle \quad . \tag{4}$$

Let $J_{i,j} \in \{0, 1\}$ be the elements of the $N \times N$ adjacency matrix J that describes the neighbouring structure of the N individuals. We consider that the individuals live on a regular lattice, where each corner has the same number Q of edges. The time evolution of the probability $p\,(\eta, t) = p\,(\eta_1, \ldots, \eta_N, t)$ is given by the master equation

$$\frac{d}{dt}p\left(\eta,t\right)=\sum_{i=1}^{N}w_{\eta_i,1-\eta_i}p\left(\eta_1,...,1-\eta_i,...,\eta_N,t\right)-\sum_{i=1}^{N}w_{1-\eta_i,\eta_i}p\left(\eta_1,...,\eta_i,...,\eta_N,t\right),$$

for $\eta_i\in\{0,1\}$ and with the transition rates

$$w_{\eta_i,1-\eta_i}=\beta\left(\sum_{j=1}^{N}J_{ij}\eta_j\right)\eta_i+\alpha\left(1-\eta_i\right)\text{ and }w_{1-\eta_i,\eta_i}=\beta\left(\sum_{j=1}^{N}J_{ij}\eta_j\right)(1-\eta_i)+\alpha\eta_i.$$

Now we will use the vector representation given by

$$\begin{aligned}\left|\psi\left(t\right)\right\rangle&=\sum_{\eta_1=0}^{1}\sum_{\eta_2=0}^{1}...\sum_{\eta_N=0}^{1}p\left(\eta_1,...,\eta_N,t\right)\left(c_1^{+}\right)^{\eta_1}...\left(c_N^{+}\right)^{\eta_N}\left|O\right\rangle\\&=\sum_{\eta}p\left(\eta,t\right)\prod_{i=1}^{N}\left(c_i^{+}\right)^{\eta_i}\left|O\right\rangle\quad,\end{aligned}\tag{5}$$

where $|O\rangle$ represents the vacuum state and $|\eta\rangle$ represents the configuration of the lattice. Hence, the time evolution of the state vector $|\psi(t)\rangle$ is given by

$$\frac{d}{dt}\left|\psi\left(t\right)\right\rangle=L\left|\psi\left(t\right)\right\rangle\quad,\tag{6}$$

where the evolution operator L can be written in terms of the creation and annihilation operators, after some calculations from the master equation, as

$$L=\alpha W_0+\lambda V\quad,\tag{7}$$

with $\lambda=\beta Q$. From now on we will consider only 1 dimensional lattices. Hence, each individual has $Q=2$ neighbours.

Definition 2.2. Let B_i, Q_i and n_i be the local operators given by

$$B_i=\left(\mathbb{1}-c_i^{+}\right)c_i\quad,\tag{8}$$

$$Q_i=\frac{1}{2}\left(\mathbb{1}-c_i\right)c_i^{+}\quad,\tag{9}$$

$$n_i=c_i^{+}c_i\quad.\tag{10}$$

The operator n_i is called the number operator. In the following Lemma, we observe the result of applying these operators to the state variables.

LEMMA 2.3. *The local operators presented in Definition 2.2 applied to the state* *variables* $|0\rangle$ *and* $|1\rangle$ *give:*

$$B_i\,|0\rangle = 0 \quad \textit{and} \quad B_i\,|1\rangle = |0\rangle - |1\rangle \quad , \tag{11}$$

$$Q_i\,|0\rangle = \frac{1}{2}\left(|1\rangle - |0\rangle\right) \quad \textit{and} \quad Q_i\,|1\rangle = 0 \quad , \tag{12}$$

$$n_i\,|0\rangle = 0 \quad \textit{and} \quad n_i\,|1\rangle = |1\rangle \quad . \tag{13}$$

The proof of this Lemma is trivial and it will not be presented here.

2.2 The σ representation

We start to observe that the operator B_i has eigenvalues 0 and -1 associated to the right eigenvectors $|0\rangle$ and $|1\rangle - |0\rangle$. So, it is convenient to change the coordinated system to these right eigenvectors that we define by

$$\left|\tilde{0}\right\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \left|\tilde{1}\right\rangle = |1\rangle - |0\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad . \tag{14}$$

The left eigenvectors of B_i are $\left\langle\tilde{0}\right| = \langle 0| + \langle 1|$ and $\left\langle\tilde{1}\right| = \langle 1|$, associated with 0 and -1 respectively.

LEMMA 2.4. *The local operators presented in Definition 2.2 applied to the state variables* $\left|\tilde{0}\right\rangle$ *and* $\left|\tilde{1}\right\rangle$ *give*

$$B_i\left|\tilde{0}\right\rangle = 0 \quad \textit{and} \quad B_i\left|\tilde{1}\right\rangle = -\left|\tilde{1}\right\rangle \quad , \tag{15}$$

$$Q_i\left|\tilde{0}\right\rangle = \frac{1}{2}\left|\tilde{1}\right\rangle \quad \textit{and} \quad Q_i\left|\tilde{1}\right\rangle = -\frac{1}{2}\left|\tilde{1}\right\rangle \quad , \tag{16}$$

$$n_i\left|\tilde{0}\right\rangle = 0 \quad \textit{and} \quad n_i\left|\tilde{1}\right\rangle = \left|\tilde{1}\right\rangle + \left|\tilde{0}\right\rangle \quad . \tag{17}$$

Once again the proof of this Lemma is trivial. Let the state of two sites of the lattice be denoted by e.g.

$$\left|\tilde{0}\tilde{0}\right\rangle = \left|\tilde{0}\right\rangle \otimes \left|\tilde{0}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad . \tag{18}$$

Similarly, we define the two sites states $\left|\tilde{0}\tilde{1}\right\rangle$, $\left|\tilde{1}\tilde{0}\right\rangle$ and $\left|\tilde{1}\tilde{1}\right\rangle$. The operators that act on these two sites are given by, e.g., $B_1 + B_2$, where $B_1 = B_i \otimes \mathbb{1}$ acts on the first site and $B_2 = \mathbb{1} \otimes B_i$ acts on the second site. Therefore, $B_1 + B_2$ acts, e.g., on the pair $\left|\tilde{0}\tilde{1}\right\rangle$ giving

$$\begin{aligned}
(B_1 + B_2)\left|\tilde{0}\tilde{1}\right\rangle &= (B_i \otimes \mathbb{1})\left(\left|\tilde{0}\right\rangle \otimes \left|\tilde{1}\right\rangle\right) + (\mathbb{1} \otimes B_i)\left(\left|\tilde{0}\right\rangle \otimes \left|\tilde{1}\right\rangle\right) \\
&= \left(B_i\left|\tilde{0}\right\rangle\right) \otimes \left(\mathbb{1}\left|\tilde{1}\right\rangle\right) + \left(\mathbb{1}\left|\tilde{0}\right\rangle\right) \otimes \left(B_i\left|\tilde{1}\right\rangle\right) \\
&= 0 \otimes \left|\tilde{1}\right\rangle + \left|\tilde{0}\right\rangle \otimes \left(-\left|\tilde{1}\right\rangle\right) \\
&= -\left|\tilde{0}\tilde{1}\right\rangle \quad .
\end{aligned} \tag{19}$$

In the same way we obtain that

$$(B_1 + B_2)\left|\tilde{0}\tilde{0}\right\rangle = 0 \quad , \tag{20}$$

$$(B_1 + B_2)\left|\tilde{1}\tilde{0}\right\rangle = -\left|\tilde{1}\tilde{0}\right\rangle \quad , \tag{21}$$

$$(B_1 + B_2)\left|\tilde{1}\tilde{1}\right\rangle = -2\left|\tilde{1}\tilde{1}\right\rangle \quad . \tag{22}$$

This representation can be generalized for all the sites in the lattice by

$$|\sigma\rangle = |\sigma_1\sigma_2 \,...\, \sigma_N\rangle \ , \quad \sigma_i \in \{\tilde{0}, \tilde{1}\} \quad , \tag{23}$$

which we call the σ representation. Let $\hat{B} = B_i$ be the operator defined in Eq. (2.2). To act on the $|\sigma\rangle$ vector we define the operators

$$W_0 = \sum_{i=1}^{N} B_i \quad , \tag{24}$$

where $B_i = \mathbb{1}^{\otimes(i-1)} \otimes \hat{B} \otimes \mathbb{1}^{\otimes(N-i)}$. Hence, generalizing the calculations presented in Eq. (19), we have that

$$\begin{aligned} W_0\,|\sigma\rangle &= \sum_{i=1}^{N} B_i\,|\sigma_1\sigma_2 \,...\, \sigma_N\rangle \\ &= -\sum_{i=1}^{N} \sigma_i\,|\sigma_1\sigma_2 \,...\, \sigma_N\rangle \quad . \end{aligned} \tag{25}$$

Therefore, the operator $W_0 = \sum_{i=1}^{N} B_i$ has eigenvalues given by

$$\Lambda(\sigma) = -\sum_{i=1}^{N} \sigma_i \quad . \tag{26}$$

Let $\hat{Q} = Q_i$ and $\hat{n} = n_i$ be the operators defined in Eq. (2.2). To operate in the two sites states we define the operators

$$Q_1 = \hat{Q} \otimes \mathbb{1}, \quad Q_2 = \mathbb{1} \otimes \hat{Q}, \quad n_1 = \hat{n} \otimes \mathbb{1} \quad \text{and} \quad n_2 = \mathbb{1} \otimes \hat{n}\ . \tag{27}$$

Theorem 2.5. *With the operators defined in Eq.* (27) *the following rules are satisfied*

$$(Q_1 n_2 + n_1 Q_2)\left|\tilde{0}\tilde{0}\right\rangle = 0 \quad , \tag{28}$$

$$(Q_1 n_2 + n_1 Q_2)\left|\tilde{0}\tilde{1}\right\rangle = \frac{1}{2}\left|\tilde{1}\tilde{0}\right\rangle + \frac{1}{2}\left|\tilde{1}\tilde{1}\right\rangle \quad , \tag{29}$$

$$(Q_1 n_2 + n_1 Q_2)\left|\tilde{1}\tilde{0}\right\rangle = \frac{1}{2}\left|\tilde{0}\tilde{1}\right\rangle + \frac{1}{2}\left|\tilde{1}\tilde{1}\right\rangle \quad , \tag{30}$$

$$(Q_1 n_2 + n_1 Q_2)\left|\tilde{1}\tilde{1}\right\rangle = -\frac{1}{2}\left|\tilde{0}\tilde{1}\right\rangle - \frac{1}{2}\left|\tilde{1}\tilde{0}\right\rangle - \left|\tilde{1}\tilde{1}\right\rangle \quad . \tag{31}$$

Proof. Due to the similarity of the calculations, here we prove only the second rule of the theorem. We start to observe that

$$\begin{aligned} Q_1 n_2 &= \left(\hat{Q} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \hat{n}\right) \\ &= \left(\hat{Q}\mathbb{1}\right) \otimes \left(\mathbb{1}\hat{n}\right) \\ &= \hat{Q} \otimes \hat{n} \quad , \end{aligned} \tag{32}$$

and, in the same way, we have $n_1 Q_2 = \hat{n} \otimes \hat{Q}$. Hence, applying the Lemma 2.4, we obtain that

$$\begin{aligned} Q_1 n_2 \left|\tilde{0}\tilde{1}\right\rangle &= \left(\hat{Q} \otimes \hat{n}\right)\left(\left|\tilde{0}\right\rangle \otimes \left|\tilde{1}\right\rangle\right) \\ &= \left(\hat{Q}\left|\tilde{0}\right\rangle\right) \otimes \left(\hat{n}\left|\tilde{1}\right\rangle\right) \\ &= \frac{1}{2}\left|\tilde{1}\right\rangle \otimes \left(\left|\tilde{1}\right\rangle + \left|\tilde{0}\right\rangle\right) \\ &= \frac{1}{2}\left|\tilde{1}\tilde{1}\right\rangle + \frac{1}{2}\left|\tilde{1}\tilde{0}\right\rangle \quad , \end{aligned} \tag{33}$$

and

$$\begin{aligned} n_1 Q_2 \left|\tilde{0}\tilde{1}\right\rangle &= \left(\hat{n} \otimes \hat{Q}\right)\left(\left|\tilde{0}\right\rangle \otimes \left|\tilde{1}\right\rangle\right) \\ &= \left(\hat{n}\left|\tilde{0}\right\rangle\right) \otimes \left(\hat{Q}\left|\tilde{1}\right\rangle\right) \\ &= 0 \otimes \left(-\frac{1}{2}\left|\tilde{1}\right\rangle\right) \\ &= 0 \quad . \end{aligned} \tag{34}$$

Therefore, summing Eqs. (33) and (34) follows that $\left(Q_1 n_2 + n_1 Q_2\right)\left|\tilde{0}\tilde{1}\right\rangle = \frac{1}{2}\left|\tilde{1}\tilde{0}\right\rangle + \frac{1}{2}\left|\tilde{1}\tilde{1}\right\rangle$. ☐

3. Series expansion

We observe that the annihilation and creation operators that appear in the evolution operator L presented in Eq. (7) are given (see [4, 11]) by the expressions

$$W_0 = \sum_{i=1}^{N} B_i \quad \text{and} \quad V = \sum_{i=1}^{N} Q_i \left(n_{i-1} + n_{i+1}\right) \quad , \tag{35}$$

where the V operator can be reorganized and written in the form

$$V = \sum_{i=1}^{N} \left(Q_i n_{i+1} + n_i Q_{i+1}\right) \quad . \tag{36}$$

From Eq. (25) we observe that

$$\sum_{i=1}^{N} B_i \left|O\right\rangle = \sum_{i=1}^{N} B_i \left|\tilde{0} \ldots \tilde{0}\right\rangle = -\sum_{i=1}^{N} 0 \left|\tilde{0} \ldots \tilde{0}\right\rangle = 0 \quad , \tag{37}$$

and from Eq. (28) we also observe that

$$\sum_{i=1}^{N} \left(Q_i n_{i+1} + n_i Q_{i+1}\right) |O\rangle = 0 \quad . \tag{38}$$

Therefore, we conclude that $|O\rangle = \left|\tilde{0} \ldots \tilde{0}\right\rangle$ is an eigenvector of L for the zero eigenvalue $L\,|O\rangle = \alpha 0 + \lambda 0 = 0$. From Eq. (25) we also observe that the subdominant eigenvalue of W_0 is -1 and the correspondent eigenvector is $|\psi_0\rangle = \left|.\tilde{1}.\right\rangle$

$$W_0\,|\psi_0\rangle = \sum_{i=1}^{N} B_i \left|.\tilde{1}\right\rangle = -\left|.\tilde{1}.\right\rangle = -\,|\psi_0\rangle \quad . \tag{39}$$

Now, we are interested in determining the subdominant eigenvalue of the evolution operator L. Let $|\psi\rangle$ denote this subdominant eigenvector and A the correspondent eigenvalue

$$L\,|\psi\rangle = A\,|\psi\rangle \quad . \tag{40}$$

Hence, the gap between the dominant eigenvalue and the subdominant eigenvalue of L is given by

$$\Gamma = 0 \ - A = -A \quad . \tag{41}$$

We assume that $|\psi\rangle$ and A can be expanded in powers of the creation rate λ

$$|\psi\rangle = |\psi_0\rangle + \lambda\,|\psi_1\rangle + \lambda^2\,|\psi_2\rangle + \ \ldots = \sum_{n=0}^{\infty} \lambda^n\,|\psi_n\rangle \quad , \tag{42}$$

$$A = A_0 + \lambda A_1 + \lambda^2 A_2 + \ \ldots = \sum_{n=0}^{\infty} A_n \lambda^n \quad , \tag{43}$$

where $|\psi_0\rangle = \left|.\tilde{1}.\right\rangle$ and $A_0 = -1$. Therefore, the expansion of the gap between the dominant and the subdominant eigenvalues Γ is given by

$$\Gamma = 1 \ - \lambda A_1 - \lambda^2 A_2 - \ \ldots \quad . \tag{44}$$

We choose the vectors $|\psi_n\rangle$ to be orthogonal to the vector $|\psi_0\rangle$

$$\langle\psi_0|\psi_n\rangle = 0\,, \quad \forall n \geq 1 \quad , \tag{45}$$

and we show how to do it in the following computations. Inserting the expanded expressions of $|\psi\rangle$ and A in Eq. (40) we obtain

$$\begin{aligned}
&(\alpha W_0 + \lambda V)\sum_{n=0}^{\infty}\lambda^n\left|\psi_n\right\rangle = \left(\sum_{m=0}^{\infty}A_m\lambda^m\right)\left(\sum_{n=0}^{\infty}\lambda^n\left|\psi_n\right\rangle\right)\\
\Leftrightarrow &\sum_{n=0}^{\infty}W_0\left|\psi_n\right\rangle\lambda^n + \sum_{n=0}^{\infty}V\left|\psi_n\right\rangle\lambda^{n+1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A_m\left|\psi_n\right\rangle\lambda^{m+n}\\
\Leftrightarrow &\sum_{n=0}^{\infty}W_0\left|\psi_n\right\rangle\lambda^n + \sum_{n=1}^{\infty}V\left|\psi_{n-1}\right\rangle\lambda^{n} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}A_m\left|\psi_{n-m}\right\rangle\lambda^{n}\\
\Leftrightarrow &W_0\left|\psi_0\right\rangle + \sum_{n=1}^{\infty}\left(W_0\left|\psi_n\right\rangle + V\left|\psi_{n-1}\right\rangle\right)\lambda^n =\\
&\qquad A_0\left|\psi_0\right\rangle + \sum_{n=1}^{\infty}\sum_{m=0}^{n}A_m\left|\psi_{n-m}\right\rangle\lambda^n\ . \qquad (46)
\end{aligned}$$

Comparing the same powers of λ in both sides of Eq. (46) we verify that

$$W_0\left|\psi_0\right\rangle = A_0\left|\psi_0\right\rangle \quad , \qquad (47)$$

in the case of $n = 0$ and for other values

$$W_0\left|\psi_n\right\rangle + V\left|\psi_{n-1}\right\rangle = \sum_{m=0}^{n}A_m\left|\psi_{n-m}\right\rangle \quad . \qquad (48)$$

Multiplying by $\langle\psi_0|$ and using the orthogonality $\langle\psi_0|\psi_n\rangle$, $\forall n \neq 0$, we obtain

$$\begin{aligned}
&\langle\psi_0|W_0|\psi_n\rangle + \langle\psi_0|V|\psi_{n-1}\rangle = \sum_{m=0}^{n}A_m\left\langle\psi_0|\psi_{n-m}\right\rangle\\
\Leftrightarrow &A_0\left\langle\psi_0|\psi_n\right\rangle + \langle\psi_0|V|\psi_{n-1}\rangle = \sum_{m=0}^{n}A_m\delta_{0,n-m}\\
\Leftrightarrow &A_0 0 + \langle\psi_0|V|\psi_{n-1}\rangle = A_n \quad . \qquad (49)
\end{aligned}$$

Hence, the coefficients A_n of the expansion of the subdominant eigenvalue of L can be computed recursively by the formula

$$A_n = \langle\psi_0|V|\psi_{n-1}\rangle\ ,\ \forall n \geq 1\ , \quad \text{and} \quad A_0 = -1 \quad . \qquad (50)$$

Now, we have to determine the vectors $|\psi_n\rangle$. Since W_0 has eigenvalues given by $\Lambda(\sigma) = -\sum_{i=1}^{N}\sigma_i$ (see Eqs. (25) and (26)) this operator can be written by the expression

$$\begin{aligned} W_0 &= \underbrace{\sum_{\sigma_1=\tilde{0}}^{1} \ldots \sum_{\sigma_N=\tilde{0}}^{1}}_{\sigma_1+\ldots+\sigma_N\neq 0} \left|\sigma_1 \ldots \sigma_N\right\rangle \Lambda\left(\sigma_1 \ldots \sigma_N\right) \left\langle\sigma_1 \ldots \sigma_N\right| \\ &= \sum_{\sigma}{}' \left|\sigma\right\rangle \Lambda\left(\sigma\right) \left\langle\sigma\right| \quad , \end{aligned} \tag{51}$$

and will be denoted by $\hat{W}_0$ from now on. Let $\hat{R}$ be the operator given by

$$\begin{aligned} \hat{R} &= \underbrace{\sum_{\sigma_1=\tilde{0}}^{1} \ldots \sum_{\sigma_N=\tilde{0}}^{1}}_{\sigma_1+\ldots+\sigma_N\notin\{0,-1\}} \left|\sigma_1 \ldots \sigma_N\right\rangle \frac{1}{\Lambda\left(\sigma_1 \ldots \sigma_N\right)-A_0} \left\langle\sigma_1 \ldots \sigma_N\right| \\ &= \sum_{\sigma}{}'' \left|\sigma\right\rangle \frac{1}{\Lambda\left(\sigma\right)-A_0} \left\langle\sigma\right| \quad . \end{aligned} \tag{52}$$

From Eq. (48), we observe that

$$\begin{aligned} W_0 \left|\psi_n\right\rangle + V\left|\psi_{n-1}\right\rangle &= A_0\left|\psi_n\right\rangle + \sum_{m=1}^{n} A_m \left|\psi_{n-m}\right\rangle \\ \Leftrightarrow \left(W_0 - A_0\right)\left|\psi_n\right\rangle &= -V\left|\psi_{n-1}\right\rangle + \sum_{m=1}^{n} A_m \left|\psi_{n-m}\right\rangle \quad , \end{aligned} \tag{53}$$

and applying $\hat{R}$ we obtain

$$\hat{R}\left(W_0 - A_0\right)\left|\psi_n\right\rangle = -\hat{R}V\left|\psi_{n-1}\right\rangle + \sum_{m=1}^{n} A_m \hat{R}\left|\psi_{n-m}\right\rangle \quad . \tag{54}$$

But

$$\begin{aligned} \hat{R}\left(\hat{W}_0 - A_0\right) &= \sum_{\sigma}{}'' \left|\sigma\right\rangle \frac{1}{\Lambda\left(\sigma\right)-A_0} \left\langle\sigma\right| \left(\hat{W}_0 - A_0\right) \\ &= \sum_{\sigma}{}'' \left|\sigma\right\rangle \frac{1}{\Lambda\left(\sigma\right)-A_0} \left(\Lambda\left(\sigma\right)\left\langle\sigma\right| - A_0\left\langle\sigma\right|\right) \\ &= \sum_{\sigma}{}'' \left|\sigma\right\rangle \left\langle\sigma\right| \quad , \end{aligned} \tag{55}$$

and joying to this sum the terms for the cases $\sigma_1 + \ldots + \sigma_N = 0$ and $\sigma_1 + \ldots + \sigma_N = -1$ we complete the eigenbasis $\sum_\sigma \left|\sigma\right\rangle \left\langle\sigma\right| = \mathbb{1}$. Hence,

$$\hat{R}\left(\hat{W}_0 - A_0\right) = \sum_{\sigma} \left|\sigma\right\rangle \left\langle\sigma\right| - \left|O\right\rangle \left\langle O\right| - \left|\psi_0\right\rangle \left\langle\psi_0\right| \quad , \tag{56}$$

and therefore,

$$\hat{R}\left(\hat{W}_0 - A_0\right)\left|\psi_n\right\rangle = \mathbb{1}\left|\psi_n\right\rangle - \left|O\right\rangle\left\langle O|\psi_n\right\rangle - \left|\psi_0\right\rangle\left\langle\psi_0|\psi_n\right\rangle = \left|\psi_n\right\rangle \quad . \tag{57}$$

Hence, from Eq. (54) we obtain that

$$|\psi_n\rangle = -\hat{R}V\,|\psi_{n-1}\rangle + \sum_{m=1}^{n-1} A_m \hat{R}\,|\psi_{n-m}\rangle, \quad \forall n \geq 2 \quad . \tag{58}$$

3.1 Explicit calculation of series expansion

We will now compute explicitly some coefficients of the expansion of the gap between the dominant and the subdominant eigenvector of the evolution operator for the SIS model. Here, we do not consider the translation invariance of the lattice, in contrast with de Oliveira [1]. For the initial state

$$|\psi_0\rangle = |\tilde{0}\tilde{1}\tilde{0}\rangle \tag{59}$$

and unperturbed eigenvalue $A_0 = -1$ we obtain

$$A_1 = \langle\psi_0|V|\psi_0\rangle = 0 \quad . \tag{60}$$

This value result from

$$\begin{aligned} V|\psi_0\rangle &= \Big(\,(Q_1 n_2 + n_1 Q_2) + (Q_2 n_3 + n_2 Q_3) + (Q_3 n_1 + n_3 Q_1)\Big)\,|\tilde{0}\tilde{1}\tilde{0}\rangle \\ &= \frac{1}{2}\Big(\,|\tilde{1}\tilde{0}\tilde{0}\rangle + |\tilde{1}\tilde{1}\tilde{0}\rangle + |\tilde{0}\tilde{0}\tilde{1}\rangle + |\tilde{0}\tilde{1}\tilde{1}\rangle\Big) \end{aligned} \tag{61}$$

and therefore

$$A_1 = \langle\tilde{0}\tilde{1}\tilde{0}|V|\psi_0\rangle = 0 \tag{62}$$

since $\langle\tilde{0}\tilde{1}\tilde{0}|\tilde{1}\tilde{1}\tilde{0}\rangle = 0$ etc. due to the orthonormality of the states. The state vector of first order in the series expansion is given by

$$\begin{aligned} |\psi_1\rangle &= -RV|\psi_0\rangle \\ &= -\sum_{\sigma}{}''|\sigma\rangle\,\frac{1}{\Lambda\,(\sigma) - A_0}\,\langle\sigma|\cdot V|\psi_0\rangle \quad . \end{aligned} \tag{63}$$

Since

$$\begin{aligned} \sum_{\sigma}{}''|\sigma\rangle\,\frac{1}{\Lambda\,(\sigma) - A_0}\,\langle\sigma| = \quad & |\tilde{1}\tilde{1}\tilde{0}\rangle\frac{1}{-2+1}\langle\tilde{1}\tilde{1}\tilde{0}| + |\tilde{1}\tilde{0}\tilde{1}\rangle\frac{1}{-2+1}\langle\tilde{1}\tilde{0}\tilde{1}| \\ & + |\tilde{0}\tilde{1}\tilde{1}\rangle\frac{1}{-2+1}\langle\tilde{0}\tilde{1}\tilde{1}| + |\tilde{1}\tilde{1}\tilde{1}\rangle\frac{1}{-3+1}\langle\tilde{1}\tilde{1}\tilde{1}| \quad , \end{aligned}$$

we have from Eq. (63)

$$\begin{aligned} |\psi_1\rangle &= -\frac{1}{2}\left(\frac{1}{-2+1}|\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{-2+1}|\tilde{0}\tilde{1}\tilde{1}\rangle\right) \\ &= \frac{1}{2}\left(|\tilde{1}\tilde{1}\tilde{0}\rangle + |\tilde{0}\tilde{1}\tilde{1}\rangle\right) \quad . \end{aligned} \tag{64}$$

For the next terms in the expansion we have to start with system size $N = 5$, hence starting state

$$|\psi_0\rangle = |\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle \tag{65}$$

then because of $(\hat{Q}_i\hat{n}_{i+1} + \hat{n}_i\hat{Q}_{i+1})|\tilde{0}\tilde{0}\rangle = 0$ we obtain as before

$$A_1 = 0 \quad , \quad |\psi_1\rangle = \frac{1}{2}\left(|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + |\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle\right) \tag{66}$$

and computing $V|\psi_1\rangle$ as in Eq. (61) we obtain now

$$A_2 = \langle\psi_0|V|\psi_1\rangle = -\frac{1}{2} \quad . \tag{67}$$

The second state $|\psi_2\rangle$ of the series expansion

$$|\psi_2\rangle = -RV|\psi_1\rangle + A_1R|\psi_0\rangle \tag{68}$$

gives, with $A_1 = 0$, explicitly

$$\begin{aligned}|\psi_2\rangle = {} & \frac{1}{4}|\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{8}|\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle - \frac{1}{2}|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{2}|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle \\ & + \frac{1}{4}|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle - \frac{1}{2}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{4}|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\rangle + \frac{1}{8}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\rangle\end{aligned} \tag{69}$$

and the third coefficient gives

$$A_3 = \langle\psi_0|V|\psi_2\rangle = \frac{1}{2} \quad . \tag{70}$$

With the previous coefficients A_i and the vectors $|\psi_i\rangle$, we obtain for

$$|\psi_3\rangle = -RV|\psi_2\rangle + A_2R|\psi_1\rangle \tag{71}$$

the explicit expression given by

$$\begin{aligned}|\psi_3\rangle = {} & \frac{1}{4}\,|\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{1}{16}\,|\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{15}{16}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle - \frac{1}{8}\,|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle \\ & + \frac{1}{16}\,|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{3}{8}\,|\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{8}\,|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{32}\,|\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle \\ & + \frac{1}{48}\,|\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle - \frac{3}{8}\,|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{5}{32}\,|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{16}\,|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle \\ & - \frac{3}{4}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle - \frac{1}{4}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{15}{16}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{3}{8}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\rangle \\ & + \frac{5}{32}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{8}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle + \frac{1}{16}\,|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle - \frac{3}{8}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle \\ & - \frac{1}{8}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{16}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{8}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\rangle + \frac{1}{16}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\rangle \\ & + \frac{1}{32}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\rangle + \frac{1}{48}\,|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\rangle \quad .\end{aligned}$$

Then for the A_4 coefficient we obtain

$$A_4 = \langle \psi_0 | V | \psi_3 \rangle = -\frac{15}{16} \quad . \tag{72}$$

In Table 1, we present a few more coefficients A_n computed in an elementary computer. We observe that the computation of the c_n coefficient involves a square matrix of size 2^{2n+1}. Hence, the next coefficients can be computed only on more sophisticated computers, with a higher memory capacity. Using many more coefficients, we could apply a Padé analyses and obtain probably the best critical threshold and critical exponent values of the SIS model.

n	$c_n = -A_n$
0	1
1	0
2	0.5
3	-0.5
4	0.9375
5	-1.8125
6	3.94010416666667
7	-8.79687500000000
8	20.45668764467593
9	-48.49340518904322

Table 1. The coefficients c_n of the expansion of the gap $\Gamma = \sum_{n=0}^{\infty} -A_n \lambda^n$.

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References

[1] M.J. de Oliveira, *Perturbative series expansion for the gap of the evolution operator associated with the contact process*, Phys. Rev. E 74 (2006), pp. 041121.

[2] R. Dickman, *Nonequilibrium lattice models: Series analysis of steady states*, J. Stat. Phys. 55 (1989), pp. 997–1026.

[3] M. Doi, *Stochastic theory of diffusion-controlled reactions*, J. Phys. A 9 (1976), pp. 1479–1495.

[4] P. Grassberger, A. de la Torre, *Reggeon Field Theory (Schlögel's First Model) on a Lattice: Monte Carlo Calculations of Critical Behaviour*, Ann. Phys. 122 (1979), pp. 373–396.

[5] P. Grassberger, M. Scheunert, *Fock-space methods for identical classical objects*, Fortschritte der Physik 28 (1980), pp. 547–578.

[6] H. Hinrichsen, *Nonequilibrium Critical Phenomena and Phase Transitions into Absorbing States*, Adv. Phys. 49 (2000), pp. 815–958 (also available in arxiv: cond-mat/0001070v2).

[7] I. Jensen, R. Dickman, *Time-dependent perturbation theory for nonequilibrium lattice models*, J. Stat. Phys. 71 (1993), pp. 89–127.

[8] J. Martins, A. Pinto, N. Stollenwerk, *A scaling analysis in the SIRI epidemiological model*, Journal of Biological Dynamics (2009), pp. 1–21 (*Accepted for publication*).

[9] L. Peliti, *Path integral approach to birth-death processes on a lattice*, J. Physique 46 (1985), pp. 1469–1483.

[10] N. Stollenwerk, J. Martins, A. Pinto, *The phase transition lines in pair approximation for the basic reinfection model SIRI*, Phys. Lett. A 371 (2007), pp. 379–388.

[11] N. Stollenwerk, M. Aguiar, *The SIRI stochastic model with creation and annihilation operators*, arXiv:0806.4565v1 (2008), pp. 1–10.

[12] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam, 1992.